# A Logic of Sequentiallity

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# Overview

- We will present a logic that can be used to model (among other things) higher order imperative programs.

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- We will present a logic that can be used to model (among other things) higher order imperative programs.
- Based on *game semantics* — a flexible approach to modelling programs and logics (initially PCF and MLL, but extended to much more, such as higher order imperative programs.)

# A simple games model

- Our work involves a simple games model originally developed in [AJ94]
- Despite its simplicity, one can embed the richer structure of the later games models using e.g. *exponentials* [HHM07]
- Thus this model can be considered a “foundational framework” for game semantics.

# Our work here

- We present a *syntactic representation* of this foundational semantic framework.
- Thus an "assembly language" for a number of logics and programming languuages.
- For example, in this system we can embed:
  - Multiplicative linear logic
  - Imperative languages.

# Games

## Definition

A *game* is a triple $(M_A, \lambda_A, P_A)$ where

- $M_A$ is a set of moves
- $\lambda_A : M_A \to \{O, P\}$ divides these moves into *player*-moves and *opponent*-moves
- $M_A^{\circledast}$ is the set of $\lambda_A$-alternating sequences over $M_A$ (we do not require that O starts)
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In practice we deal with *positive* games (where P always starts) or *negative* games (where O always starts.)

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If $A$ and $B$ are negative (positive) so are $A \| B$ and $A + B$, and so on...
We can also take **Lifts** of games, adding an extra move at the beginning sending positive games to negative games, and vice versa.
We have an operation $-^{\perp}$ inverting P and O moves.

# Strategies

**Definition**

A *strategy* $\sigma$ for a game $(M_A, \lambda_A, P_A)$ is a nonempty subset of $P_A$ such that

- All plays in $\sigma$ end in a P-move
- If $sop \in \sigma$ then $s \in \sigma$
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A strategy is *total* if P can always respond to valid O-moves.

# Example?

- There are $\omega^2$ total strategies on $\mathbf{N} + \mathbf{N}$ corresponding to how P corresponds to the left and right initial questions:

| $\mathbb{N}$ | $+$ | $\mathbb{N}$ | |
|---|---|---|---|
| | | $q$ | O |
| | | 7 | P |

| $\mathbb{N}$ | $+$ | $\mathbb{N}$ | |
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| 5 | | | P |

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$$\begin{array}{ccccc} \mathbb{N} & + & \mathbb{N} & \\ q & & & \mathrm{O} \\ 5 & & & \mathrm{P} \end{array}$$

- There are many more strategies in $\mathbf{N} \| \mathbf{N}$ as *both* initial questions can be posed, in either order.

# Categorical Structure

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- We have a category $\mathcal{G}$ of negative games where an arrow $A \to B$ is a strategy on $B \| A^{\perp}$.
  - Symmetric monoidal closed with products, using merges and sums.
- We can also construct a category of positive games which is dual to $\mathcal{G}$, but can also be seen as a subcategory of $\mathcal{G}$.
- We can equip these categories with various (co)monads, embedding notions of *justification structure* and *innocence* used in the full abstraction results of PCF.

# Formulae of WS

- We now describe a (polarised) logic representing this foundational structure.

Positive and negative formulae:

$$P := \mathbf{0} \mid \downarrow N \mid P \parr Q \mid P \oplus Q \mid P \lhd Q \mid P \oslash N$$

$$N := \mathbf{1} \mid \uparrow P \mid N \otimes M \mid M \& N \mid N \oslash M \mid N \lhd P$$

- **Idea:** Positive (negative) formulae represent positive (negative) games. For each polarity, we have formulae representing the empty game, lifts, merge, sum, and leftmerge respectively.

# Proof system

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- Proofs in this logic are in *bijective correspondence* with total strategies on the appropriate game.
- The rules are heavily focussed — for each sequent (=formula) $A$, rules are either determined or correspond to Player chosing a move to play at a point in the strategy.
- Each formula can be decomposed into a *focus* and a *context* and the proof rules depend on the focus of the formula — a form of *deep inference*

# Contexts and Foci

We have a notion of *focussing context*
$S\{\_\} := \{\_\} \mid S\{\_\} \oslash N \mid S\{\_\} \lhd P$

- Each formula can be uniquely decomposed into something of the form $S\{A\}$ where $S$ is a focussing context and $A$ is a *focus* — a formula that does not have a left-merge ($\lhd$ or $\oslash$) at the top level.

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- A formula $A \lhd (B_1 \parr \ldots \parr B_n)$ corresponds to the linear sequent $B_1^\perp, \ldots, B_n^\perp \vdash A$.

# Proof Rules 1

$$\frac{}{\Gamma \vdash S\{\mathbf{1}\}}$$

$$\frac{\Gamma \vdash S\{N \oslash M\} \qquad \Gamma \vdash S\{M \oslash N\}}{\Gamma \vdash S\{M \otimes N\}}$$

$$\frac{\Gamma \vdash S\{P\}}{\Gamma \vdash S\{P \oplus Q\}}$$

$$\frac{\Gamma \vdash S\{P \lhd Q\}}{\Gamma \vdash S\{P \parr Q\}}$$

$$\frac{\Gamma \vdash S\{Q\}}{\Gamma \vdash S\{P \oplus Q\}}$$

$$\frac{\Gamma \vdash S\{Q \lhd P\}}{\Gamma \vdash S\{P \parr Q\}}$$

$$\frac{\Gamma \vdash S\{M\} \qquad \Gamma \vdash S\{N\}}{\Gamma \vdash S\{M \& N\}}$$

# Proof Rules 2

$$\frac{\Gamma \vdash P}{\Gamma \vdash \uparrow P}$$

$$\frac{\Gamma \vdash N}{\Gamma \vdash \downarrow N}$$

$$\frac{\Gamma \vdash S\{\uparrow (P \parr Q)\}}{\Gamma \vdash S\{\uparrow P \lhd Q\}}$$

$$\frac{\Gamma \vdash S\{\downarrow (M \otimes N)\}}{\Gamma \vdash S\{\downarrow M \oslash N\}}$$

$$\frac{\Gamma \vdash S\{\uparrow (P \oslash N)\}}{\Gamma \vdash S\{\uparrow P \oslash N\}}$$

$$\frac{\Gamma \vdash S\{\downarrow (N \lhd P)\}}{\Gamma \vdash S\{\downarrow N \lhd P\}}$$

# Semantic Correspondence

- We give semantics of proofs as total strategies — a proof of $A$ is represented as a total strategy on $[\![A]\!]$.

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- We have "full completeness" — every total strategy is the denotation of some proof, and this correspondence is bijective.
- Thus an "assembly language" for total strategies.

# Example

Let $\mathbf{B} = \uparrow(\downarrow \mathbf{1} \oplus \downarrow \mathbf{1})$, the standard game of Booleans. There are two possible total strategies on $\mathbf{B}$ — one that answers the initial question with a final answer $\downarrow \mathbf{1}$ on the left, or on the right. These correspond to the proofs below for $i \in \{1, 2\}$:

$$\cfrac{\cfrac{\cfrac{\cfrac{}{\vdash \mathbf{1}}}{\vdash \downarrow \mathbf{1}}}{\vdash \downarrow \mathbf{1} \oplus \downarrow \mathbf{1}} \text{ using rule } \oplus_i}{\vdash \uparrow(\downarrow \mathbf{1} \oplus \downarrow \mathbf{1})}$$

# Conservative Extensions

We can extend this logic by:

- Weakening the constraint of *local alternation*
- Introducing exponential rules representing the *Lamarche* exponentials on games used for e.g. SPCF
  - Using this we can embed full *Polarised Linear Logic* LLP in our system.

# Imperative Objects

We can also represent *finitary imperative objects* (e.g. a read/write Boolean with finitary lifetime) as proofs in this logic.

# Cut Elimination

We know that we can *compose strategies* — a key to game semantics of programming languages. It follows from full completeness that we have cut elimination for WS.

$$\frac{\vdash M \lhd \Gamma^{\perp} \qquad \vdash N \lhd \Delta^{\perp}}{\vdash M \lhd (\Gamma \otimes \Delta)^{\perp}}$$

But we can also describe a cut elimination procedure *syntactically*, giving an operational semantics for our "assembly language" of proofs, programs and strategies.

# Propositional Variables

- A major missing element from our system is a notion of *variables*. It would be good to introduce propositional variables representing arbitrary games.
- Thus, we would have a proof of $A \multimap A = A \lhd A^{\perp}$ representing an arbitrary copycat strategy, and so on.
- We are currently working on developing an extension to WS of this nature, and investigating its properties.

# Further Directions

Other issues at hand include

- Considering the *categorical structure* required to (fully abstractly?) model our logic.
- Term representations of proofs (something similar to CCS?) and formalising reduction etc using these ideas.
- Representing infinite games, e.g. using other infinitary exponentials (e.g. the original exponential on games representing spawning a limitless number of threads.)

# Questions?